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A CLASS OF FINSLER METRICS WITH ISOTROPIC S-CURVATURE

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ABSTRACT

In this paper, we study a class of Finsler metrics defined by a Riemannian metric and a 1-form. We characterize these metrics with isotropic S-curvature.

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1. Introduction

The S-curvature $\mathbf{S} = S(x,y)$ is an important non-Riemannian quantity in Finsler geometry which was first introduced by the second author when he studied volume comparison in Riemann-Finsler geometry [12]. The second author proved that the Bishop-Gromov volume comparison holds for Finsler manifolds with vanishing S-curvature. He also proved that the S-curvature and the Ricci curvature determine the local behavior of the Busemann-Hausdorff measure of small metric balls around a point [14]. Recent study shows that the S-curvature plays a very important role in Finsler geometry (cf., [9],[15],[17]). It interacts with the flag curvature in a mysterious way. The flag curvature $\mathbf{K} = K(P,y)$ is a Riemannian quantity which is a natural extension of the sectional curvature in Riemannian geometry. The first problem is to understand Finsler metrics whose flag curvature is independent of P containing a tangent vector y, i.e., $\mathbf{K} = K(x,y)$ is a scalar function on the tangent bundle. For Riemannian metrics of scalar flag curvature, $\mathbf{K} = K(x)$ is independent of tangent vector y at each point x. For Finsler metrics of scalar flag curvature, the flag curvature can take a very general form. It is known that, for a Finsler metric $F = F(x,y)$ of scalar flag curvature, if the S-curvature is almost isotropic, i.e.,

$$\mathbf{S} = (n+1)cF + \eta, \tag{1}$$

where $c = c(x)$ is a scalar function and η is a closed 1-form (if $\eta = 0$, then the S-curvature is said to be isotropic, see Definition 3.1), then the flag curvature must be in the following form

$$\mathbf{K} = \frac{3\tilde{c}_{x^m} y^m}{F} + \sigma, \tag{2}$$

where $\sigma = \sigma(x)$ and $\tilde{c} = \tilde{c}(x)$ are scalar functions with $c - \tilde{c} = constant$ [4]. This shows that S-curvature has impact on the flag curvature of Finsler metrics. Therefore it is an important problem to study and characterize Finsler metrics of (almost) isotropic S-curvature.

Throughout the paper, our index conventions are

$$1 \le i, j, k, \ldots \le n, \quad 2 \le A, B, C, \ldots \le n.$$

In Finsler geometry, there is an important class of Finsler metrics which were introduced and studied by G. Randers, hence are called **Randers metrics**. A Randers metric on a manifold M is a Finsler metric of the form $F = \alpha + \beta$, where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form with $\|\beta_x\|_\alpha < 1$. Let

$$r_{ij} = \frac{1}{2}(b_{i|j} + b_{j|i}), \qquad s_{ij} = \frac{1}{2}(b_{i|j} - b_{j|i}),$$
$$r_j := b^i r_{ij}, \qquad s_j := b^i s_{ij},$$

where $b_{i|j}$ denote the covariant derivatives of β with respect to α. In [5], we proved that the Randers metric $F = \alpha + \beta$ has isotropic S-curvature, $\mathbf{S} = (n+1)c(x)F$, if and only if

$$r_{ij} + b_i s_j + b_j s_i = 2c(a_{ij} - b_i b_j). \tag{3}$$

See [1] and [20] for related work. A Finsler metric on a manifold M in the following form is said to be of **Randers type**,

$$F = k_1\sqrt{\alpha^2 + k_2\beta^2} + k_3\beta, \tag{4}$$

where α is a Riemannian metric, β is a 1-form on M, $k_1 > 0, k_2$ and $k_3 \neq 0$ are constants. Clearly, Finsler metrics of Randers type are essentially Randers metrics. By a simple argument, one can prove the following

THEOREM 1.1: *For a Finsler metric of Randers type, $F = k_1\sqrt{\alpha^2 + k_2\beta^2} + k_3\beta$, it is of isotropic S-curvature, $\mathbf{S} = (n+1)cF$ if and only if β satisfies*

$$r_{ij} + \tau(s_i b_j + s_j b_i) = \frac{2c(1 + k_2 b^2)k_1^2}{k_3}(a_{ij} - \tau b_i b_j), \tag{5}$$

where

$$b := \|\beta_x\|_\alpha, \quad \tau := k_3^2/k_1^2 - k_2.$$

Among Finsler metrics of Randers type, there are nontrivial projectively flat (α, β)-metrics with constant flag curvature [11].

If a Randers metric is of scalar flag curvature, then 1 and 2 are actually equivalent [7], [19]. In fact, if a Randers metric is of constant flag curvature, then it must be of constant S-curvature [1], [2]. We have classified Randers metrics of scalar flag curvature and isotropic S-curvature [4], [7]. Further, we have characterized the locally projectively flat Finsler metrics with isotropic S-curvature [6].

It is natural to consider general Finsler metrics defined by a Riemannian metric $\alpha = \sqrt{a_{ij}y^i y^j}$ and a 1-form $\beta = b_i y^i$ with $\|\beta_x\|_\alpha < b_o$. They are expressed in the form $F = \alpha\phi(s), s = \beta/\alpha$, where $\phi(s)$ is a C^∞ positive function on $(-b_o, b_o)$. It is known that $F = \alpha\phi(\beta/\alpha)$ is a (positive definite) Finsler metric for any α and β with $\|\beta_x\|_\alpha < b_o$ if and only if ϕ satisfies the following condition (cf., [18] and [16]):

$$\phi(s) - s\phi'(s) + (\rho^2 - s^2)\phi''(s) > 0, \quad |s| \le \rho < b_o. \tag{6}$$

Such a metric is called an **(α,β)-metric**. Clearly, Finsler metrics of Randers type are special (α,β)-metrics defined by $\phi = k_1\sqrt{1+k_2 s^2} + k_3 s$.

For a positive C^∞ function $\phi = \phi(s)$ on $(-b_o, b_o)$ and a number $b \in [0, b_o)$, let

$$\Phi := -(Q - sQ')\{n\Delta + 1 + sQ\} - (b^2 - s^2)(1 + sQ)Q'',$$

where $\Delta := 1 + sQ + (b^2 - s^2)Q'$ and $Q := \phi'/(\phi - s\phi')$. In this paper, we prove the following

THEOREM 1.2: *Let $F = \alpha\phi(s), s = \beta/\alpha$, be an (α,β)-metric on a manifold and $b := \|\beta_x\|_\alpha$. Suppose that $\phi \ne k_1\sqrt{1+k_2 s^2} + k_3 s$ for any constants $k_1 > 0, k_2$ and k_3. Then F is of isotropic S-curvature, $\mathbf{S} = (n+1)cF$, if and only if one of the following holds*

(i) *β satisfies*

$$r_j + s_j = 0 \tag{7}$$

and $\phi = \phi(s)$ satisfies

$$\Phi = 0. \tag{8}$$

In this case, $\mathbf{S} = 0$.

(ii) *β satisfies*

$$r_{ij} = \epsilon\Big\{b^2 a_{ij} - b_i b_j\Big\}, \quad s_j = 0, \tag{9}$$

where $\epsilon = \epsilon(x)$ is a scalar function, and $\phi = \phi(s)$ satisfies

$$\Phi = -2(n+1)k\frac{\phi\Delta^2}{b^2 - s^2}, \tag{10}$$

where k is a constant. In this case, $\mathbf{S} = (n+1)cF$ with $c = k\epsilon$.

(iii) *β satisfies*

$$r_{ij} = 0, \quad s_j = 0. \tag{11}$$

In this case, $\mathbf{S} = 0$, regardless of the choice of a particular ϕ.

It is easy to see that (11) implies (9), while (9) implies (7). The condition (7) is equivalent to $b := \|\beta_x\|_\alpha = constant$. See Lemma 3.3 below. Thus (8) and (10) are independent of $x \in M$. Any solution $\phi = \phi(s)$ of (10) is regular on $(-b, b)$, but it might be singular at $s = \pm b$.

The mean Landsberg curvature $\mathbf{J}$ is another important non-Riemannian quantity. It has been proved that for an (α, β)-metric $F = \alpha\phi(\beta/\alpha)$, if β has constant length and ϕ satisfies (8), then F is a weakly Landsberg metric, i.e., $\mathbf{J} = 0$. See [10].

We have the following two interesting examples.

Example 1.1: Let $F = \alpha + \beta$ be the family of Randers metrics on S^3 constructed in [3] (see also [14]). It is shown that $r_{ij} = 0$ and $s_j = 0$. Thus for any C^∞ positive function $\phi = \phi(s)$ satisfying (6), the (α, β)-metric $F = \alpha\phi(\beta/\alpha)$ has vanishing S-curvature.

Example 1.2: Let $F = \alpha\phi(\beta/\alpha)$ be an (α, β)-metric defined on an open subset in R^3. If, at a point $\mathbf{x} = (x, y, z) \in R^3$ and in the direction $\mathbf{y} = (u, v, w) \in T_{\mathbf{x}}R^3$, $\alpha = \alpha(\mathbf{x}, \mathbf{y})$ and $\beta = \beta(\mathbf{x}, \mathbf{y})$ are given by

$$\alpha := \sqrt{u^2 + e^{2x}(v^2 + w^2)},$$
$$\beta := u,$$

then β satisfies (9) with $\epsilon = 1$, $b = 1$. Thus if $\phi = \phi(s)$ satisfies (10) for some constant k, then $F = \alpha\phi(\beta/\alpha)$ is of constant S-curvature $\mathbf{S} = (n+1)cF$. However, we can not find an explicit solution to (10) with $k \neq 0$.

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2. Volume forms

The S-curvature is associated with a volume form. There are two important volume forms in Finsler geometry. One is the Busemann-Hausdorff volume form and the other is the Holmes-Thompson volume form.

The Busemann-Hausdorff volume form $dV_{BH} = \sigma_{BH}(x)dx$ is given by

$$\sigma_{BH}(x) = \frac{\omega_n}{\mathrm{Vol}\left\{(y^i) \in R^n \colon F\left(x, y^i \frac{\partial}{\partial x^i}\right) < 1\right\}}$$

and the Holmes-Thompson volume form $dV_{HT} = \sigma_{HT}(x)dx$ is given by

$$\sigma_{HT}(x) = \frac{1}{\omega_n} \int_{\left\{(y^i)\in R^n | F\left(x, y^i \frac{\partial}{\partial x^i}\right)<1\right\}} \det(g_{ij})dy.$$

Here Vol denotes the Euclidean volume and

$$\omega_n := \mathrm{Vol}(\mathrm{B}^n(1)) = \frac{1}{n}\mathrm{Vol}(\mathrm{S}^{n-1}) = \frac{1}{n}\mathrm{Vol}(\mathrm{S}^{n-2}) \int_0^\pi \sin^{n-2}(t)dt$$

denotes the Euclidean volume of the unit ball in R^n. When $F = \sqrt{g_{ij}(x)y^iy^j}$ is a Riemannian metric, both volume forms are reduced to the same Riemannian volume form

$$dV_{BH} = dV_{HT} = \sqrt{\det(g_{ij}(x))}dx.$$

For an (α, β)-metric, we have the following formulas for the volume forms dV_{BH} and dV_{HT}.

Proposition 2.1: *Let $F = \alpha\phi(s), s = \beta/\alpha$ be an (α, β)-metric on an n-dimensional manifold M and $b := \|\beta_x\|_\alpha$. Let $dV = dV_{BH}$ or dV_{HT}. Let*

$$f(b) := \begin{cases} \dfrac{\int_0^\pi \sin^{n-2} t dt}{\int_0^\pi \frac{\sin^{n-2} t}{\phi(b\cos t)^n} dt} & \text{if } dV = dV_{BH} \\[2ex] \dfrac{\int_0^\pi (\sin^{n-2} t) T(b\cos t) dt}{\int_0^\pi \sin^{n-2} t dt} & \text{if } dV = dV_{HT}, \end{cases}$$

where $T(s) := \phi(\phi - s\phi')^{n-2}[(\phi - s\phi') + (b^2 - s^2)\phi'']$ and $b := \|\beta_x\|_\alpha$. Then the volume form dV is given by

$$dV = f(b)dV_\alpha,$$

where $dV_\alpha = \sqrt{\det(a_{ij})}dx$ denotes the Riemannian volume form of α.

Proof. In a coordinate system, the determinant of $g_{ij} := \frac{1}{2}[F^2]_{y^iy^j}$ is given by (cf., [16])

$$\det(g_{ij}) = \phi^{n+1}(\phi - s\phi')^{n-2}[(\phi - s\phi') + (b^2 - s^2)\phi''] \det(a_{ij}).$$

First we take an orthonormal basis at x with respect to α so that

$$\alpha = \sqrt{\sum (y^i)^2}, \quad \beta = by^1,$$

where $b = \|\beta_x\|_\alpha$. Then the volume form $dV_\alpha = \sigma_\alpha dx$ at x is given by

$$\sigma_\alpha = \sqrt{\det(a_{ij})} = 1.$$

In order to evaluate the integrals

$$\mathrm{Vol}\Big\{(y^i)\in R^n\colon F\Big(x,y^i\frac{\partial}{\partial x^i}\Big)<1\Big\}=\int_{F(x,y)<1}dy=\int_{\alpha\phi(\beta/\alpha)<1}dy$$

and

$$\int_{F(x,y)<1}\det(g_{ij})dy=\int_{\alpha\phi(\beta/\alpha)<1}\det(g_{ij})dy,$$

we take the following coordinate transformation, $\psi:(s,u^A)\to(y^i)$:

$$y^1=\frac{s}{\sqrt{b^2-s^2}}\bar{\alpha},\quad y^A=u^A, \tag{12}$$

where $\bar{\alpha}=\sqrt{\sum_{A=2}^n(u^A)^2}$. Then

$$\alpha=\frac{b}{\sqrt{b^2-s^2}}\bar{\alpha},\quad \beta=\frac{bs}{\sqrt{b^2-s^2}}\bar{\alpha}. \tag{13}$$

Thus

$$F=\alpha\phi(\beta/\alpha)=\frac{b\phi(s)}{\sqrt{b^2-s^2}}\bar{\alpha}$$

and the Jacobian of the transformation $\psi:(s,u^A)\to(y^i)$ is given by

$$\frac{b^2}{(b^2-s^2)^{3/2}}\bar{\alpha}.$$

Then

$$\begin{aligned}\mathrm{Vol}\{(y^i)\in R^n\colon F(x,y)<1\}&=\int_{\frac{b\phi(s)}{\sqrt{b^2-s^2}}\bar{\alpha}<1}\frac{b^2}{(b^2-s^2)^{3/2}}\bar{\alpha}dsdu\\&=\int_{-b}^{b}\frac{b^2}{(b^2-s^2)^{3/2}}\Big[\int_{\bar{\alpha}<\frac{\sqrt{b^2-s^2}}{b\phi(s)}}\bar{\alpha}du\Big]ds\\&=\frac{1}{n}\mathrm{Vol}(\mathrm{S}^{n-2})\int_{-b}^{b}\frac{b^2}{(b^2-s^2)^{3/2}}\Big(\frac{\sqrt{b^2-s^2}}{b\phi(s)}\Big)^n ds\\&=\frac{1}{n}\mathrm{Vol}(\mathrm{S}^{n-2})\int_{-b}^{b}\frac{(b^2-s^2)^{(n-3)/2}}{b^{n-2}\phi(s)^n}ds\\&=\frac{1}{n}\mathrm{Vol}(\mathrm{S}^{n-2})\int_0^{\pi}\frac{\sin^{n-2}t}{\phi(b\cos t)^n}dt\quad(s=b\cos t).\end{aligned}$$

Therefore

$$\sigma_{BH}=\frac{\int_0^\pi\sin^{n-2}tdt}{\int_0^\pi\frac{\sin^{n-2}t}{\phi(b\cos t)^n}dt}. \tag{14}$$

Let

$$T(s) := \phi(\phi - s\phi')^{n-2}[(\phi - s\phi') + (b^2 - s^2)\phi'']. \tag{15}$$

Then

$$\det(g_{ij}) = \phi(s)^n T(s) \det(a_{ij}).$$

By a similar argument, we get

$$\begin{aligned}\sigma_{HT} &= \frac{1}{\omega_n} \int_{F(x,y)<1} \phi(s)^n T(s) dy^1 \cdots dy^n \\ &= \frac{1}{n\omega_n} \mathrm{Vol}(\mathrm{S}^{n-2}) \int_{-b}^{b} \frac{b^2}{(b^2 - s^2)^{3/2}} \Big(\frac{\sqrt{b^2 - s^2}}{b} \Big)^n T(s) ds \\ &= \frac{\int_0^\pi (\sin^{n-2} t) T(b \cos t) dt}{\int_0^\pi \sin^{n-2} t dt}.\end{aligned}$$

Thus

$$\sigma_{HT} = \frac{\int_0^\pi (\sin^{n-2} t) T(b \cos t) dt}{\int_0^\pi \sin^{n-2} t dt}. \tag{16}$$

The above formulas for σ_{BH} and σ_{HT} are given in a special coordinate system at x and $\sigma_\alpha = 1$. Thus $dV = f(b) dV_\alpha$. This proves the proposition. ∎

Note that if $b = constant$, then $f(b) = constant$. In this case, both dV_{BH} and dV_{HT} are constant multiples of dV_α.

It is surprising to see that $dV_{HT} = dV_\alpha$ for certain functions ϕ.

Corollary 2.2: *Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be an (α, β)-metric on an n-dimensional manifold M. Let $T = T(s)$ be defined in* (15). *Suppose that $T(s) - 1$ is an odd function of s. Then $dV_{HT} = dV_\alpha$.*

Proof: Let $h(s) = T(s) - 1$. By assumption $h(-s) = -h(s)$. It is easy to see that

$$\int_0^\pi (\sin^{n-2} t) h(b \cos t) dt = 0.$$

Thus

$$\int_0^\pi (\sin^{n-2} t) T(b \cos t) dt = \int_0^\pi \sin^{n-2} t dt.$$

This implies that $\sigma_{HT} = 1$ in the above special coordinate system at x. Then in a general coordinate system $\sigma_{HT} = \sigma_\alpha$. ∎

If $\phi = 1 + s$, then $T = 1 + s$ and $T(s) - 1$ is an odd function of s. Then for a Randers metric, $dV_{HT} = dV_\alpha$. This fact is known to Y. B. Shen.

3. The S-Curvature

In this section, we are going to find a formula for the S-curvature of an (α, β)-metric on an n-dimensional manifold M.

Let $F = F(x, y)$ be a Finsler metric on an n-dimensional manifold M. There is a notion of distortion $\tau = \tau(x, y)$ on TM associated with a volume form $dV = \sigma(x)dx$.

$$\tau(x, y) = \ln \frac{\sqrt{\det(g_{ij}(x, y))}}{\sigma(x)}.$$

In fact, $\tau(x, y)$ depends only on $F_x := F|_{T_xM}$ on T_xM at each point x. Moreover, $\tau(x, y) = \tau(x)$ at a point $x \in M$ if and only if F_x is Euclidean on T_xM. The S-curvature is defined by

$$\mathbf{S}(x, y) = \frac{d}{dt}\Big[\tau\Big(c(t), \dot{c}(t)\Big)\Big]\Big|_{t=0},$$

where $c(t)$ is the geodesic with $c(0) = x$ and $\dot{c}(0) = y$ [14]. From the definition, we see that the S-curvature $\mathbf{S}(y)$ measures the rate of change in the distortion on (T_xM, F_x) in the direction $y \in T_xM$.

Let $G = y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i}$ denote the spray of F and $dV = \sigma(x)dx$ be a volume form on M. The spray coefficients G^i are defined by

$$G^i = \frac{1}{4} g^{il}\big\{[F^2]_{x^j y^l} y^j - [F^2]_{x^l}\big\}.$$

Then the S-curvature (with respect to dV) is given by

$$\mathbf{S} = \frac{\partial G^m}{\partial y^m} - y^m \frac{\partial}{\partial x^m}(\ln \sigma).$$

An important property is that $\mathbf{S} = 0$ for Berwald spaces with respect to the Busemann-Hausdorff volume form dV_{BH} [12], [13]. This explains why we choose the Busemann-Hausdorff volume form to define the S-curvature of a Finsler metric.

Definition 3.1: Let F be a Finsler metric on an n-dimensional manifold M. Let $\mathbf{S}$ denote the S-curvature of F with respect to dV_{BH}.

(a) F is of almost isotropic S-curvature if

$$\mathbf{S} = (n + 1)cF + \eta,$$

where $c = c(x)$ is a scalar function and η is a closed 1-form;

(b) F is of isotropic S-curvature if $c = c(x)$ is a scalar function and $\eta = 0$;

(c) F is of constant S-curvature if c is a constant and $\eta = 0$.

Now we compute the S-curvature of an (α,β)-metric on a manifold. Let

$$F=\alpha\phi(s), \quad s=\beta/\alpha.$$

Let $\bar{G}^i$ denote the spray coefficients of α. We have the following formula for the spray coefficients G^i of F (cf., [8], [16]):

$$G^i=\bar{G}^i+\alpha Q s^i{}_0+\Theta\{-2Q\alpha s_0+r_{00}\}\frac{y^i}{\alpha}+\Psi\{-2Q\alpha s_0+r_{00}\}b^i,$$

where $s^i{}_j:=a^{ih}s_{hj}, s^i{}_0:=s^i{}_jy^j, s_0:=s_iy^i, r_{00}:=r_{ij}y^iy^j$ and

$$Q:=\frac{\phi'}{\phi-s\phi'}, \quad \Theta=\frac{Q-sQ'}{2\Delta}, \quad \Psi=\frac{Q'}{2\Delta}, \tag{17}$$

where $\Delta:=1+sQ+(b^2-s^2)Q'$.

It is easy to see that if $\phi=\phi(s)$ satisfies

$$b^2Q+s=0,$$

then

$$\phi=a_0\sqrt{b^2-s^2},$$

where a_0 is a number independent of s.

Lemma 3.2: *If $\phi=\phi(s)$ satisfies*

$$\Psi=constant,$$

then

$$\phi=k_1\sqrt{1+k_2s^2}+k_3s,$$

where k_1,k_2 and k_3 are numbers independent of s.

To compute the S-curvature, we need the following identities:

$$\frac{\partial s}{\partial y^m}=\frac{1}{\alpha}\Big\{b_m-s\frac{y_m}{\alpha}\Big\},$$
$$\frac{\partial \alpha}{\partial y^m}=\frac{y_m}{\alpha},$$
$$\frac{\partial \bar{G}^m}{\partial y^m}=y^m\frac{\partial}{\partial x^m}\Big(\ln\sigma_\alpha\Big),$$

where $y_m:=a_{mj}y^j$. Using the above identities, we obtain

$$\frac{\partial G^m}{\partial y^m}=y^m\frac{\partial}{\partial x^m}(\ln\sigma_\alpha)+2\Psi(r_0+s_0)-\alpha^{-1}\frac{\Phi}{2\Delta^2}(r_{00}-2\alpha Q s_0),$$

where

$$\Phi := -(Q - sQ')\{n\Delta + 1 + sQ\} - (b^2 - s^2)(1 + sQ)Q''. \tag{18}$$

By Proposition 2.1, $dV = \sigma dx = f(b)\sigma_\alpha dx$. Thus

$$y^m \frac{\partial}{\partial x^m}(\ln \sigma) = \frac{f'(b)}{f(b)} y^m \frac{\partial b}{\partial x^m} + y^m \frac{\partial}{\partial x^m}(\ln \sigma_\alpha).$$

$$y^m \frac{\partial b}{\partial x^m} = \frac{b^i b_{i|m} y^m}{b} = \frac{r_0 + s_0}{b}. \tag{19}$$

Then the S-curvature is given by

$$\mathbf{S} = \Big\{ 2\Psi - \frac{f'(b)}{bf(b)} \Big\} (r_0 + s_0) - \alpha^{-1} \frac{\Phi}{2\Delta^2}(r_{00} - 2\alpha Q s_0). \tag{20}$$

Lemma 3.3: *Let β be a 1-form on a Riemannian manifold (M, α). Then $b(x) := \|\beta_x\|_\alpha = constnt$ if and only if β satisfies the following equation:*

$$r_j + s_j = 0. \tag{21}$$

Proof. This follows immediately from (19). ∎

In the case when $b = constant$, the S-curvature is given by

$$\mathbf{S} = -\alpha^{-1} \frac{\Phi}{2\Delta^2}(r_{00} - 2\alpha Q s_0). \tag{22}$$

We can prove the following

Proposition 3.4: *Let $F = \alpha\phi(\beta/\alpha)$ be an (α, β)-metric on an n-manifold. If β and ϕ satisfy conditions in Theorem 1.2 (i) or (ii) or (iii), then F has isotropic S-curvature.*

Proof. If β satisfies (7) and ϕ satisfies (8), then it follows from (20) that $\mathbf{S} = 0$.

If β satisfies (9), then

$$r_{00} = \epsilon(b^2 - s^2)\alpha^2, \quad r_0 = 0, \quad s_0 = 0.$$

By (10) and the above equations, we get from (20) that

$$\mathbf{S} = -\alpha\epsilon(b^2 - s^2)\frac{\Phi}{2\Delta^2} = (n+1)k\epsilon\alpha\phi = (n+1)k\epsilon F.$$

If β satisfies (11), then

$$r_{00} = 0, \quad r_0 = 0, \quad s_0 = 0.$$

It follows from (20) that $\mathbf{S} = 0$. ∎

To prove the necessary conditions in Theorem 1.2, we consider an (α,β)-metric $F=\alpha\phi(\beta/\alpha)$ with isotropic S-curvature, $\mathbf{S}=(n+1)cF$. By (20), the equation $\mathbf{S}=(n+1)cF$ is equivalent to the following equation:

$$\alpha^{-1}\frac{\Phi}{2\Delta^2}(r_{00}-2\alpha Q s_0)-2\Psi(r_0+s_0)=-(n+1)cF+\theta, \tag{23}$$

where

$$\theta:=-\frac{f'(b)}{bf(b)}(r_0+s_0). \tag{24}$$

To simplify the equation (23), we choose special coordinates. Fix an arbitrary point x. Take a local coordinate system at x as in (12). We have

$$r_1=br_{11}, \qquad r_A=br_{1A},$$
$$s_1=0, \qquad s_A=bs_{1A}.$$

Let

$$\bar r_{10}:=\sum_{A=2}^n r_{1A}y^A, \quad \bar s_{10}:=\sum_{A=2}^n s_{1A}y^A \quad \bar r_{00}:=\sum_{A,B=2}^n r_{AB}y^Ay^B,$$
$$\bar r_0:=\sum_{A=2}^n r_Ay^A \quad \bar s_0:=\sum_{A=2}^n s_Ay^A.$$

Let $\theta=t_iy^i$. Then t_i are given by

$$t_1=-\frac{f'(b)}{f(b)}r_{11}, \quad t_A=-\frac{f'(b)}{f(b)}(r_{1A}+s_{1A}). \tag{25}$$

From (12), we have

$$r_0=\frac{sbr_{11}}{\sqrt{b^2-s^2}}\bar\alpha+b\bar r_{10}, \quad s_0=\bar s_0=b\bar s_{10} \tag{26}$$

and

$$r_{00}=\frac{s^2\bar\alpha^2}{b^2-s^2}r_{11}+2\frac{s\bar\alpha}{\sqrt{b^2-s^2}}\bar r_{10}+\bar r_{00}. \tag{27}$$

Substituting (26) and (27) into (23) and by use of (13), we find that (23) is equivalent to the following two equations:

$$\frac{\Phi}{2\Delta^2}(b^2-s^2)\bar r_{00}=-\Big\{s\Big(\frac{s\Phi}{2\Delta^2}-2\Psi b^2\Big)r_{11}+(n+1)cb^2\phi-sbt_1\Big\}\bar\alpha^2, \tag{28}$$

$$\Big(\frac{s\Phi}{\Delta^2}-2\Psi b^2\Big)(r_{1A}+s_{1A})-(b^2Q+s)\frac{\Phi}{\Delta^2}s_{1A}-bt_A=0. \tag{29}$$

Let

$$\Upsilon := \left[\frac{s\Phi}{\Delta^2} - 2\Psi b^2\right]'.$$

We see that $\Upsilon = 0$ if and only if

$$\frac{s\Phi}{\Delta^2} - 2\Psi b^2 = b^2\mu,$$

where $\mu = \mu(x)$ is independent of s. We divide the problem into three cases: (i) $\Phi = 0$, (ii) $\Phi \neq 0$, $\Upsilon = 0$ and (iii) $\Phi \neq 0$, $\Upsilon \neq 0$.

4. $\Phi = 0$

In this section, we study the simplest case when $\Phi = 0$.

PROPOSITION 4.1: *Let $F = \alpha\phi(\beta/\alpha)$ be an (α, β)-metric. Assume that $\Phi = 0$ but $\phi \neq k_1\sqrt{1 + k_2 s^2} + k_3 s$ for any constants $k_1 > 0, k_2$ and k_3. If F has isotropic S-curvature, then*

$$r_0 + s_0 = 0.$$

In this case, $\mathbf{S} = 0$.

Proof. Take a special coordinate system at x as in (12). (28) and (29) are reduced to

$$s\left\{\frac{f'(b)}{bf(b)} - 2\Psi\right\}b^2 r_{11} + (n+1)cb^2\phi = 0 \tag{30}$$

$$\left\{\frac{f'(b)}{bf(b)} - 2\Psi\right\}b^2(r_{1A} + s_{1A}) = 0. \tag{31}$$

Letting $s = 0$ in (30) yields

$$c = 0$$

and

$$\left\{\frac{f'(b)}{bf(b)} - 2\Psi\right\}r_{11} = 0. \tag{32}$$

If

$$\frac{f'(b)}{bf(b)} - 2\Psi = 0,$$

then, by Lemma 3.2, we have

$$\phi = k_1\sqrt{1 + k_2 s^2} + k_3 s,$$

where $k_1 > 0$, k_2 and k_3 are numbers independent of s. But this is impossible by assumption. Thus

$$\frac{f'(b)}{bf(b)} - 2\Psi \neq 0.$$

¿From (30) and (31), we conclude that

$$r_{11} = 0, \quad r_{1A} + s_{1A} = 0. \qquad \blacksquare$$

5. $\Phi \neq 0$, $\Upsilon = 0$

First, note that $\Upsilon = 0$ implies that

$$\frac{s\Phi}{\Delta^2} - 2\Psi b^2 = b^2\mu, \tag{33}$$

where $\mu = \mu(x)$ is a function on M independent of s. First, we prove the following

Lemma 5.1: *Let $F = \alpha\phi(\beta/\alpha)$ be an (α,β)-metric. Assume that $\Phi \neq 0$ and $\Upsilon = 0$. If F has isotropic S-curvature, $\mathbf{S} = (n+1)cF$, then β satisfies*

$$r_{ij} = ka_{ij} - \epsilon b_i b_j + \frac{1}{b^2}(r_i b_j + r_j b_i), \tag{34}$$

where $k = k(x)$, $\epsilon = \epsilon(x)$, and $\phi = \phi(s)$ satisfies the following ODE:

$$(k - \epsilon s^2)\frac{\Phi}{2\Delta^2} = \big\{\nu + (k - \epsilon b^2)\mu\big\}s - (n+1)c\phi, \tag{35}$$

where $\nu = \nu(x)$. If $s_0 \neq 0$, then ϕ satisfies the following additional ODE:

$$\frac{\Phi}{\Delta^2}(Qb^2 + s) = b^2(\mu + \lambda), \tag{36}$$

where $\lambda = \lambda(x)$.

Proof. Since $\Phi \neq 0$ and notice that $\bar{r}_{00}$ and $\bar{\alpha}$ are independent of s, it follows from (28) and (29) that in a special coordinate system (s, y^a) at a point x,

$$r_{AB} = k\delta_{AB}, \tag{37}$$

$$s\Big(\frac{s\Phi}{2\Delta^2} - 2\Psi b^2\Big)r_{11} + (n+1)cb^2\phi + k\frac{\Phi}{2\Delta^2}(b^2 - s^2) = bst_1, \tag{38}$$

$$\Big(\frac{s\Phi}{\Delta^2} - 2\Psi b^2\Big)(r_{1A} + s_{1A}) - (b^2Q + s)\frac{\Phi}{\Delta^2}s_{1A} - bt_A = 0, \tag{39}$$

where $k = k(x)$ is independent of s.

Let

$$r_{11} = -(k - \epsilon b^2).$$

Then (34) holds. By (33), we have

$$\frac{s\Phi}{2\Delta^2} - 2\Psi b^2 = b^2\mu - \frac{s\Phi}{2\Delta^2}.$$

Then (38) and (39) become

$$b(k - \epsilon s^2)\frac{\Phi}{2\Delta^2} = st_1 + sb\mu(k - b^2\epsilon) - (n+1)cb\phi. \tag{40}$$

$$b^2\mu(r_{1A} + s_{1A}) - \frac{\Phi}{\Delta^2}(Qb^2 + s)s_{1A} - bt_A = 0. \tag{41}$$

Letting $t_1 = b\nu$ in (40) we get (35).

Suppose that $s_0 \neq 0$. Rewrite (41) as

$$\Big\{b^2\mu - \frac{\Phi}{\Delta^2}(Qb^2 + s)\Big\}s_{1A} = bt_A bb - b^2\mu r_{1A}.$$

We can see that there is a function $\lambda = \lambda(x)$ on M such that

$$\mu b^2 - \frac{\Phi}{\Delta^2}(Qb^2 + s) = -b^2\lambda.$$

This gives (36). ∎

Lemma 5.2: *Let $F = \alpha\phi(\beta/\alpha)$ be an (α, β)-metric. Assume that*

$$\phi \neq k_1\sqrt{1 + k_2 s^2} + k_3 s$$

for any constants $k_1 > 0, k_2$ and k_3. If $\Upsilon = 0$, then $b = constant$.

Proof. Suppose that $b \neq constant$. Then b can be viewed as a variable over the manifold. By assumption,

$$\frac{s\Phi}{\Delta^2} - 2\Psi b^2 = b^2\mu,$$

where $\mu = \mu(x)$. Note that $\Delta^2(\frac{s\Phi}{\Delta^2} - 2\Psi b^2 - b^2\mu)$ is a polynomial of degree six in b by (17). More precisely, we have

$$-\mu Q'^2 b^6 - \big\{Q'^2 - 2\mu Q'(1 + sQ - s^2 Q')\big\}b^4 + (\cdots)b^2 + (\cdots) = 0. \tag{42}$$

Thus

$$\mu Q'^2 = 0, \quad Q'^2 - 2\mu Q'(1 + sQ - s^2 Q') = 0.$$

Then $Q' = 0$, which implies that $\phi = 1 + cs$. This is impossible as we exclude the case $\phi = k_1\sqrt{1 + k_2 s^2} + k_3 s$. ∎

Proposition 5.3: *Let $F = \alpha\phi(\beta/\alpha)$ be an (α,β)-metric. Suppose that $b^2Q + s \neq 0$, $\Phi \neq 0$ and $\Upsilon = 0$. If F has isotropic S-curvature, $\mathbf{S} = (n+1)cF$, then*

$$r_{ij} = \epsilon(b^2 a_{ij} - b_i b_j), \quad s_j = 0, \tag{43}$$

where $\epsilon = \epsilon(x)$ is a scalar function, and $\phi = \phi(s)$ satisfies

$$\epsilon(b^2 - s^2)\Phi = -2(n+1)c\phi\Delta^2. \tag{44}$$

If $\epsilon \neq 0$, then $c/\epsilon = constant$.

Proof. First by Lemmas 5.2 and 3.3, we have

$$r_0 + s_0 = 0.$$

Then by (20), we receive

$$\mathbf{S} = -\alpha^{-1}\frac{\Phi}{2\Delta^2}\{r_{00} - 2\alpha Q s_0\}.$$

By Lemma 5.1,

$$r_{00} = (k - \epsilon s^2)\alpha^2 + \frac{2s}{b^2} r_0 \alpha.$$

Then

$$\mathbf{S} = -(k - \epsilon s^2)\frac{\Phi}{2\Delta^2}\alpha + \frac{\Phi}{b^2\Delta^2}(b^2Q + s)s_0.$$

By (35), we get

$$\mathbf{S} = -s\{\nu + (k - \epsilon b^2)\mu\}\alpha + \frac{\Phi}{b^2\Delta^2}(b^2Q + s)s_0 + (n+1)c\phi\alpha. \tag{45}$$

By our assumption, $\mathbf{S} = (n+1)cF$, we get from (45) that

$$-s\{\nu + (k - \epsilon b^2)\mu\}\alpha + \frac{\Phi}{b^2\Delta^2}(b^2Q + s)s_0 = 0. \tag{46}$$

Letting $y^i = \delta b^i$ for a sufficiently small $\delta > 0$ yields

$$-\delta\{\nu + (k - \epsilon b^2)\mu\}b^2 = 0.$$

We conclude that

$$\nu + (k - \epsilon b^2)\mu = 0. \tag{47}$$

Then (46) is reduced to

$$\frac{\Phi}{b^2\Delta^2}(b^2Q + s)s_0 = 0.$$

Since $\Phi \neq 0$ and $b^2Q + s \neq 0$, we conclude that

$$s_0 = 0.$$

Then

$$r_0 = -s_0 = 0.$$

It follows from (34) that

$$r_{ij} = ka_{ij} - \epsilon b_i b_j. \tag{48}$$

Contracting (48) with b^i gives

$$r_j = (k - \epsilon b^2) b_j = 0.$$

Since $\beta \neq 0$, we get

$$k = \epsilon b^2 \tag{49}$$

and (48) becomes

$$r_{ij} = \epsilon(b^2 a_{ij} - b_i b_j).$$

Finally, (44) follows from (35), (47) and (49).

If $\epsilon \neq 0$, then letting $s = 0$ in (44) yields that $c/\epsilon = constant$ since $b = constant$. ∎

6. $\Phi \neq 0$ and $\Upsilon \neq 0$

In this section, we shall consider the case when $\phi = \phi(s)$ satisfies

$$\Phi \neq 0, \qquad \Upsilon \neq 0. \tag{50}$$

First, we need the following

Lemma 6.1: *Let $F = \alpha\phi(s), s = \beta/\alpha$, be an (α, β)-metric on an n-dimensional manifold. Assume that $\phi = \phi(s)$ satisfies (50). Suppose that F has isotropic S-curvature, $\mathbf{S} = (n+1)cF$. Then*

$$r_{ij} = ka_{ij} - \epsilon b_i b_j - \lambda(s_i b_j + s_j b_i), \tag{51}$$

where $\lambda = \lambda(x), k = k(x)$ and $\epsilon = \epsilon(x)$ are scalar functions of x and

$$-2s(k - \epsilon b^2)\Psi + (k - \epsilon s^2)\frac{\Phi}{2\Delta^2} + (n+1)c\phi - s\nu = 0, \tag{52}$$

where

$$\nu := -\frac{f'(b)}{bf(b)}(k - \epsilon b^2). \tag{53}$$

If in addition $s_0 \neq 0$, then

$$-2\Psi - \frac{Q\Phi}{\Delta^2} - \lambda\Big(\frac{s\Phi}{\Delta^2} - 2\Psi b^2\Big) = \delta, \tag{54}$$

where

$$\delta := -\frac{f'(b)}{bf(b)}(1-\lambda b^2). \tag{55}$$

Proof. By assumption, $\Phi \not\equiv 0$. Similar to the proof of Lemma 5.1, it follows from (28) that there is a function $k = k(x)$ independent of s, such that

$$\bar{r}_{00} = k\bar{\alpha}^2, \tag{56}$$

$$s\Big(\frac{s\Phi}{2\Delta^2} - 2\Psi b^2\Big)r_{11} + (n+1)cb^2\phi + k\frac{\Phi}{2\Delta^2}(b^2 - s^2) = sbt_1. \tag{57}$$

Let

$$r_{11} = k - \epsilon b^2,$$

where $\epsilon = \epsilon(x)$ is independent of s. By (25), $t_1 = b\nu$, where ν is given by (53). Plugging them into (57) yields (52).

Suppose that $s_0 = 0$. Then

$$bs_{1A} = s_A = 0.$$

Then (29) is reduced to

$$\Big(\frac{s\Phi}{\Delta^2} - 2\Psi b^2\Big)r_{1A} - bt_A = 0. \tag{58}$$

By assumption, $\Upsilon \neq 0$, we know that $\frac{s\Phi}{\Delta^2} - 2\Psi b^2$ is not independent of s. It follows from (58) that

$$r_{1A} = 0, \quad t_A = 0.$$

The above identities together with $r_{11} = k - \epsilon b^2$ and $t_1 = b\nu$ imply the following identities

$$r_{ij} = ka_{ij} - \epsilon b_i b_j. \tag{59}$$

Suppose that $s_0 \neq 0$. Then $s_{A_o} = bs_{1A_o} \neq 0$ for some A_o.

Differentiating (29) with respect to s yields

$$\Upsilon\, r_{1A} - \Big[\frac{Q\Phi}{\Delta^2} + 2\Psi\Big]' b^2 s_{1A} = 0. \tag{60}$$

Let

$$\lambda := -\frac{r_{1A_o}}{b^2 s_{1A_o}}.$$

Plugging it into (60) yields

$$-\lambda\Upsilon - \Big[\frac{Q\Phi}{\Delta^2} + 2\Psi\Big]' = 0. \tag{61}$$

It follows from (61) that

$$\delta := -\frac{Q\Phi}{\Delta^2} - 2\Psi - \lambda\Big[\frac{s\Phi}{\Delta^2} - 2\Psi b^2\Big]$$

is a number independent of s. By assumption that $\Upsilon \neq 0$, we obtain from (60) and (61) that

$$r_{1A} + \lambda b^2 s_{1A} = 0. \tag{62}$$

(56) and (62) together with $r_{11} = k - \epsilon b^2$ imply that

$$r_{ij} + \lambda(b_i s_j + b_j s_i) = k a_{ij} - \epsilon b_i b_j. \tag{63}$$

By (25) and (62),

$$t_A = \frac{f'(b)}{f(b)}(b^2\lambda - 1)s_{1A}.$$

On the other hand, by (29) and (62), we obtain

$$b t_A = \delta b^2 s_{1A}.$$

Combining the above identities, we get (55). ∎

Lemma 6.2: *Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be an (α,β)-metric. Suppose that $\phi = \phi(s)$ satisfies* (50) *and $\phi \neq k_1\sqrt{1+k_2 s^2} + k_3 s$ for any constants $k_1 > 0$, k_2 and k_3. If F has isotropic S-curvature, then*

$$r_j + s_j = 0.$$

Proof. Suppose that $r_j + s_j \neq 0$, then $b := \|\beta_x\|_\alpha \neq$ *constant* in a neighborhood. We view b as a variable in (52) and (54). Since $\phi = \phi(s)$ is a function independent of x, (52) and (54) actually give rise infinitely many ODEs on ϕ.

First, we consider (52). Let

$$eq := \Delta^2\Big\{ -2s(k-\epsilon b^2)\Psi + (k - \epsilon s^2)\frac{\Phi}{2\Delta^2} + (n+1)c\phi - s\nu\Big\}.$$

We have

$$eq = \Xi_0 + \Xi_2 b^2 + \Xi_4 b^4,$$

where Ξ_0, Ξ_2, Ξ_4 are independent of b and

$$\Xi_4 := \{(\epsilon - \nu)s + (n+1)c\phi\}\frac{\phi^2}{(\phi - s\phi')^4}(\phi'')^2.$$

It follows from (52) that $eq = 0$. Thus

$$\Xi_0 = 0, \quad \Xi_2 = 0, \quad \Xi_4 = 0.$$

Since $\phi'' \neq 0$, the equation $\Xi_4 = 0$ is equivalent to the following ODE:

$$(\epsilon - \nu)s + (n+1)c\phi = 0.$$

we conclude that

$$\epsilon = \nu, \quad c = 0.$$

Then by a direct computation we get

$$\Xi_0 + \Xi_2 s^2 = -\frac{1}{2}(1+sQ)\{(n-1)(k-\epsilon s^2)(Q - sQ') + 2kQ + 2\epsilon s\}.$$

Then $\Xi_0 = 0$ and $\Xi_2 = 0$ imply that

$$(n-1)(k-\epsilon s^2)(Q - sQ') + 2kQ + 2\epsilon s = 0, \tag{64}$$

Suppose that $(k, \epsilon) \neq 0$. We claim that $k \neq 0$. If this is not true, i.e., $k = 0$, then $\epsilon \neq 0$ and (64) is reduced to

$$-(n-1)s(Q - sQ') + 2 = 0.$$

Letting $s = 0$, we get a contradiction.

Now we have that $k \neq 0$. It is easy to see that $Q(0) = 0$. Let

$$\tilde{Q} := Q(s) - sQ'(0).$$

Plugging it into (64) yields

$$(n-1)(k-\epsilon s^2)(\tilde{Q} - s\tilde{Q}') + 2k\tilde{Q} + 2(kQ'(0) + \epsilon)s = 0.$$

Since $\tilde{Q} = q_m s^m + o(s^m)$ where $m > 1$ is an integer, we see that $kQ'(0) + \epsilon = 0$. The above equation is reduced to

$$(n-1)(k-\epsilon s^2)(\tilde{Q} - s\tilde{Q}') + 2k\tilde{Q} = 0.$$

We obtain

$$\tilde{Q} = c_1 \frac{s^{(n+1)/(n-1)}}{(k-\epsilon s^2)^{1/(n-1)}}.$$

We must have $c_1 = 0$, that is, $\tilde{Q} = 0$. We get

$$Q(s) - sQ'(0) = 0.$$

Then it follows that

$$Q(s) = Q'(0)s.$$

In this case, $\phi = c_1\sqrt{1 + c_2 s^2}$ where $c_1 > 0$ and c_2 are numbers independent of s. This case is excluded in the assumption. Therefore, $k = 0$ and $\epsilon = 0$. Then (51) is reduced to

$$r_{ij} = -\lambda(s_j b_i + s_i b_j).$$

Then

$$r_j + s_j = (1 - \lambda b^2)s_j.$$

By the assumption at the beginning of the proof, $r_j + s_j \neq 0$, we conclude that $1 - \lambda b^2 \neq 0$ and $s_j \neq 0$. By Lemma 6.1, $\phi = \phi(s)$ satisfies (54). Let

$$EQ := \Delta^2 \Big\{ -2\Psi - \frac{Q\Phi}{\Delta^2} - \lambda\Big(\frac{s\Phi}{\Delta^2} - 2\Psi b^2\Big) - \delta \Big\}.$$

We have

$$EQ = \Omega_0 + \Omega_2 b^2 + \Omega_4 b^4,$$

where $\Omega_0, \Omega_2, \Omega_4$ are independent of b and

$$\Omega_4 = (Q')^2(\lambda - \delta).$$

By (54), $EQ = 0$. Thus

$$\Omega_0 = 0, \quad \Omega_2 = 0, \quad \Omega_4 = 0.$$

Since $Q' \neq 0$, $\Omega_4 = 0$ implies that

$$\delta = \lambda.$$

By a direct computation, we get

$$\Omega_0 + \Omega_2 s^2 = (1 + sQ)\{(n+1)Q(Q - sQ') - Q' + \lambda[ns(Q - sQ') - 1]\}.$$

The equations $\Omega_0 = 0$ and $\Omega_2 = 0$ imply that $\Omega_0 + \Omega_2 s^2 = 0$, that is,

$$(n+1)Q(Q - sQ') - Q' + \lambda[ns(Q - sQ') - 1] = 0.$$

We obtain

$$Q = -\frac{[k_0 n(n+1) - 1]\lambda s \pm \sqrt{\lambda k_0 (k_0(1+n)^2 - 1 + \lambda s^2)}}{k_0(n+1)^2 - 1}.$$

Plugging it into $\Omega_2 = 0$ yields

$$k_0 \lambda = 0.$$

Then

$$Q = \frac{\lambda s}{k_0(n+1)^2 - 1}.$$

This implies that $\phi = k_1\sqrt{1 + k_2 s^2}$ where $k_1 > 0$ and k_2 are numbers independent of s. This case is excluded in the assumption of the lemma. Therefore, $r_j + s_j = 0$. ∎

PROPOSITION 6.3: *Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be an (α, β)-metric. Suppose that $\phi = \phi(s)$ satisfies* (50) *and $\phi \neq k_1\sqrt{1 + k_2 s^2} + k_3 s$ for any constants $k_1 > 0$, k_2 and k_3. If F is of isotropic S-curvature, $\mathbf{S} = (n + 1)cF$, then*

$$r_{ij} = \epsilon(b^2 a_{ij} - b_i b_j), \quad s_j = 0, \tag{65}$$

where $\epsilon = \epsilon(x)$ is a scalar function on M and $\phi = \phi(s)$ satisfies

$$\epsilon(b^2 - s^2)\frac{\Phi}{2\Delta^2} = -(n + 1)c\phi. \tag{66}$$

Proof. Contracting (51) with b^i yields

$$r_j + s_j = (k - \epsilon b^2)b_j + (1 - \lambda b^2)s_j. \tag{67}$$

By Lemma 6.2, $r_j + s_j = 0$. It follows from (67) that

$$(1 - \lambda b^2)s_j + (k - \epsilon b^2)b_j = 0. \tag{68}$$

Contracting (68) with b^j yields

$$(k - \epsilon b^2)b^2 = 0.$$

We get

$$k = \epsilon b^2.$$

Then (51) is reduced to

$$r_{ij} = \epsilon(b^2 a_{ij} - b_i b_j) - \lambda(b_i s_j + b_j s_i).$$

By (53),

$$\nu = 0.$$

Then (52) is reduced to (66).

We claim that $s_0 = 0$. Suppose that $s_0 \neq 0$. By (68), we conclude that

$$\lambda = 1/b^2.$$

By (55),

$$\delta = 0.$$

It follows from (54) that

$$(b^2 Q + s)\Phi = 0.$$

This is impossible by the assumption $\Phi \neq 0$. ∎

7. Proof of Theorem 1.1

Notice that in Lemma 6.1, there is no restriction on ϕ other than (50). Let $\phi = k_1\sqrt{1+k_2s^2}+k_3s$, where $k_1 > 0$, k_2 and k_3 are numbers independent of s. It is easy to check that, if $k_3 \neq 0$, then ϕ satisfies (50). Let $F = \alpha\phi(\beta/\alpha)$, where α is a Riemannian metric and β is a 1-form on an n-dimensional manifold. It is easy to see that if F is a Finsler metric, then $1+k_2b^2 > 0$, where $b := \|\beta_x\|_\alpha$. By Lemma 6.1, we can easily prove Theorem 1.1.

Proof of Theorem 1.1. Assume that F is of isotropic S-curvature, $\mathbf{S} = (n+1)cF$. By Lemma 6.1, β satisfies (51) and ϕ satisfies (52) and, further, it satisfies (54) if $s_0 \neq 0$.

First, we plug $\phi = k_1\sqrt{1+k_2s^2}+k_3s$ into

$$eq := -2s(k-\epsilon b^2)\Psi + (k-\epsilon s^2)\frac{\Phi}{2\Delta^2} + (n+1)c\phi - s\nu.$$

By (52), the coefficients of the Taylor expansion of eq in s must be zero. We obtain

$$\begin{aligned} c &= \frac{k_3k}{2(1+k_2b^2)k_1^2} \\ \nu &= \Big\{\Big(\frac{n}{1+k_2b^2}+1\Big)\frac{k_3^2}{k_1^2} - k_2\Big\}k \\ \epsilon &= \Big\{\frac{k_3^2}{k_1^2} - k_2\Big\}k. \end{aligned}$$

Assume that $s_0 \neq 0$. We plug $\phi = k_1\sqrt{1+k_2s^2}+k_3s$ into

$$EQ = -2\Psi - \frac{Q\Phi}{\Delta^2} - \lambda\Big(\frac{s\Phi}{\Delta^2} - 2\Psi b^2\Big) - \delta.$$

By (54), the coefficients of the Taylor expansion of EQ in s must be zero. We obtain

$$\begin{aligned} \lambda &= \frac{k_3^2}{k_1^2} - k_2 \\ \delta &= \Big(\frac{n}{1+k_2b^2}+1\Big)\frac{k_3^2}{k_1^2} - k_2. \end{aligned}$$

This proves the necessary conditions by (51).

Conversely, if β satisfies (5), then F is of isotropic S-curvature by (20). The proof is direct, so it is omitted. ∎

References

[1] D. Bao and C. Robles, *On Randers metrics of constant curvature*, Reports on Mathematical Physics **51** (2003), 9–42.

[2] D. Bao, C. Robles and Z. Shen, *Zermelo navigation on Riemannian manifolds*, Journal of Differential Geometry **66** (2004), 377–435.

[3] D. Bao and Z. Shen, *Finsler metrics of constant flag curvature on the Lie group S^3*, Journal of the London Mathematical Society **66** (2002), 453–467.

[4] X. Chen(g), X. Mo and Z. Shen, *On the flag curvature of Finsler metrics of scalar curvature*, Journal of the London Mathematical Society **68** (2003), 762–780.

[5] X. Chen(g) and Z. Shen, *Randers metrics with special curvature properties*, Osaka Journal of Mathematics **40** (2003), 87–101.

[6] X. Cheng and Z. Shen, *Projectively flat Finsler metrics with almost isotropic S–curvature*, Acta Mathematica Scientia **26B** (2006), 307–313.

[7] X. Cheng and Z. Shen, *Randers metrics of scalar flag curvature*, 2006, preprint.

[8] S. S. Chern and Z. Shen, *Riemann–Finsler Geometry*, World Scientific Publishing Co. Pte. Ltd., Singapore, 2005.

[9] C.-W. Kim and J.-W. Yim, *Finsler manifolds with positive constant flag curvature*, Geometriae Dedicata **98** (2003), 47–56.

[10] B. Li and Z. Shen, *On a class of weak Landsberg metrics*, Science in China Series A **50** (2007), 75–85.

[11] B. Li and Z. Shen, *On a class of projectively flat Finsler metrics with constant flag curvature*, International Journal of Mathematics **18** (2007), 1–12.

[12] Z. Shen, *Volume comparison and its applications in Riemann–Finsler geometry*, Advances in Mathematics **128** (1997), 306–328.

[13] Z. Shen, *Differential Geometry of Spray and Finsler Spaces*, Kluwer Academic Publishers, Dordrecht, 2001.

[14] Z. Shen, *Lectures on Finsler Geometry*, World Scientific Co., Singapore, 2001.

[15] Z. Shen, *Finsler metrics with K=0 and S=0*, Canadian Journal of Mathematics **55** (2003), 112–132.

[16] Z. Shen, *Landsberg curvature, S-curvature, and Riemann curvature*, in *A Sampler of Finsler Geometry*, Mathematical Sciences Research Institute Publications, 50, Cambridge University Press, Cambridge, 2004, pp. 303–355.

[17] Z. Shen, *Nonpositively curved Finsler manifolds with constant S-curvature*, Mathematische Zeitschrift **249** (2005), 625–639.

[18] Z. Shen, *On a class of Landsberg metrics in Finsler geometry*, Canadian Journal of Mathematics, to appear.

[19] Z. Shen and G. C. Yildirim, *A characterization of Randers metrics of scalar flag curvature*, 2008, preprint.

[20] Z. Shen and H. Xing, *On Randers metrics with isotropic S–curvature* , Acta Mathematica Sinica, English Series **24** (2008), 789–796.